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AUTHOR Carlson, James E.
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ABSTRACT

The multivariate analysis of variance approach to the analysis of data from repeated measures designs that include covariates is reviewed and contrasted with univariate methods. Assumptions necessary for meaningful interpretation of the results of the multivariate analysis are discussed and contrasted to those of the univariate approaches. An extension of the full-rank multivariate linear model approach and its use in testing various hypotheses and assumptions is also presented. Questions of model fitting versus hypothesis testing in exploratory research are also included. Examples illustrating the various methods are provided.
(Author/JKS)

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ANALYSIS OF COVARIANCE WITH REPEATED MEASURES

MULTIVARIATE LINEAR MODEL APPROACH

James E. Carlson

University of Ottawa

April, 1980

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1.0 Introduction

The analysis of repeated measurements data through use of the multivariate linear model is discussed in several texts including Bock (1975) and Timm (1975). Timm uses the Full Rank Multivariate Linear Model (FRMLM), first discussed by Timm and Carlson (1973), whereas Bock uses an overparameterized less than full rank model. No extensive discussion of these designs with covariates could be found in the literature.

Since the Full-Rank approach is less complex mathematically than the Less-than-Full-Rank approach it is used in this paper. In the next section the FRMLM is outlined, and in Section 2 its use in Repeated Measurements designs without covariates is briefly presented. The remaining sections contain the major topic of this paper, the case of covariates in repeated measurements designs.

1.1 The Full-Rank Multivariate Linear Model (FRMLM)

The FRMLM as first outlined by Timm and Carlson (1973) is

$$(1.1) \quad Y = WU + E$$

$$R'U = \oplus \text{ (optional)}$$

where: Y = an $N \times q$ matrix of random observations of N subjects on q variates, with expectation WU

W = an $N \times p$, nonrandom, design matrix of full column rank, containing predictors in multivariate regression, group-membership indicators in MANOVA, and a combination of indicators and covariates in MANCOVA designs

U = a $p \times q$ matrix of nonrandom parameters: regression coefficients and intercepts, or group means

E = an $N \times q$ matrix of errors with rows distributed independently $N_q(0, \Sigma)$

R' = an $r \times p$ optional matrix of known coefficients expressing restrictions on the rows of U . R' is of full row rank

$\textcircled{H}$ = an $r \times q$ matrix of known constants, usually all zeros, for the restrictions

The model is referred to as either restricted or unrestricted depending on whether restrictions are or are not incorporated into the model. Recent work (Timm, 1979; Carlson & Timm, 1980) to extend the model to cover a wider class of restrictions is pertinent to the designs discussed in this paper. These restrictions are expressed in the form:

$$(1.2) \quad R'UA = \textcircled{H}$$

where A is a q by u matrix of known coefficients. The restriction in (1.1) can be considered to be a special case with $A = I_q$ the identity matrix of order q . R' is used either to place restrictions on the rows of U or to designate certain rows whose elements are subject to restrictions specified in A . These roles may also be reversed.

Hypotheses are stated in the form:

$$(1.3) \quad H: C'UA = \Gamma$$

where C' and A are $c \times p$ and $q \times u$ matrices of coefficients and Γ is a $c \times u$ matrix of constants, usually all zeros. C' must be of full row rank and A of full column rank.

With no restrictions the minimum variance unbiased linear estimator of U is

$$(1.4) \quad \hat{U} = (W'W)^{-1}W'Y$$

and the estimate of error variation about the model is

$$(1.5) \quad \hat{Q} = (Y - W\hat{U})'(Y - W\hat{U})$$

with $N-p$ degrees of freedom. When restrictions are imposed on the model the estimators are

$$(1.6) \quad \hat{U}_r = \hat{U} - (W'W)^{-1}R(R'(W'W)^{-1}R)^{-1}(R'\hat{U}A - \Theta)(A'A)^{-1}A'$$

and

$$(1.7) \quad \hat{Q}_r = \hat{Q} + (A'A)^{-1}(R'\hat{U}A - \Theta)'(R'(W'W)^{-1}R)^{-1}(R'\hat{U}A - \Theta)(A'A)^{-1}A'$$

with $N-p+r$ degrees of freedom.

When hypotheses of the form (1.3) are tested the matrix of error sums of squares and products is

$$(1.8) \quad S_e = A'\hat{Q}A \quad \text{or} \quad S_{er} = A'\hat{Q}_rA$$

while the matrix of sums of squares and products for the hypothesis is

$$(1.9) \quad S_h = (C'\hat{U}A - \Gamma)'(C'(W'W)^{-1}C)^{-1}(C'\hat{U}A - \Gamma)$$

$$\text{or} \quad S_{hr} = (C'\hat{U}A - \Gamma)'(C'F(W'W)^{-1}F'C)^{-1}(C'\hat{U}A - \Gamma)$$

$$\text{with } F = I - (W'W)^{-1}R(R'(W'W)^{-1}R)^{-1}R'$$

The various multivariate test statistics (Wilks' lambda, Roy's largest root, Pillai's Trace, Lawley-Hotelling Trace) are based on the eigenvalues, λ_1 , of the determinantal equation

$$(1.10) \quad |S_h - \lambda_1 S_e| = 0 \quad \text{or} \quad |S_{hr} - \lambda_1 S_{er}| = 0$$

(see, for example, Timm, 1975).

2.0 The Groups by Measures Design with No Covariates

To analyze data from a repeated measures design with M measures

and G groups we use (1.1) with $q = M$ and $p = G$. The columns of Y are the M measures,

$$(2.1) \quad Y = \begin{bmatrix} y_1(1) & y_2(1) & \dots & y_m(1) & \dots & y_M(1) \\ y_1(2) & y_2(2) & \dots & y_m(2) & \dots & y_M(2) \\ \vdots & \vdots & & \vdots & & \vdots \\ y_1(g) & y_2(g) & \dots & y_m(g) & \dots & y_M(g) \\ \vdots & \vdots & & \vdots & & \vdots \\ y_1(G) & y_2(G) & \dots & y_m(G) & \dots & y_M(G) \end{bmatrix}$$

where $y_m(g)$ is a vector of n_g observations in the g^{th} Group on the m^{th} measure. The Design matrix, W , contains G group-membership indicators as columns,

$$(2.2) \quad W = \begin{bmatrix} 1_{n_1} & 0_{n_1} & \dots & 0_{n_1} \\ 0_{n_2} & 1_{n_2} & \dots & 0_{n_2} \\ \vdots & \vdots & & \vdots \\ 0_{n_G} & 0_{n_G} & \dots & 1_{n_G} \end{bmatrix}$$

where 1_{n_g} is an n_g -element vector of ones and 0_{n_g} an n_g -element vector of zeros. The parameter matrix, U , contains the Group means on each Measure,

$$(2.3) \quad U = \begin{bmatrix} \mu_1(1) & \mu_2(1) & \dots & \mu_M(1) \\ \mu_1(2) & \mu_2(2) & \dots & \mu_M(2) \\ \vdots & \vdots & & \vdots \\ \mu_1(G) & \mu_2(G) & \dots & \mu_M(G) \end{bmatrix}$$

where $\mu_{m(g)}$ is the population mean of the g^{th} Group on the m^{th} Measure.

Although most analyses involve no restrictions on U so that the restriction is dropped from the model, for designs in which certain interactions are assumed zero the restriction expresses the assumption so that U is estimated subject to restrictions, and sample variation associated with the interactions assumed to be zero is pooled with the error variation to get S_{er} in (1.8).

2.1 Tests on Groups

The multivariate test of equality of Group mean vectors is

$$(2.4) \quad H_{G(\text{mult.})}: \begin{bmatrix} \mu_1(1) \\ \mu_2(1) \\ \vdots \\ \mu_M(1) \end{bmatrix} = \begin{bmatrix} \mu_1(2) \\ \mu_2(2) \\ \vdots \\ \mu_M(2) \end{bmatrix} = \dots = \begin{bmatrix} \mu_1(G) \\ \mu_2(G) \\ \vdots \\ \mu_M(G) \end{bmatrix}$$

which is expressed in the form of (1.3) by choosing $A = I_M$, $\Gamma = 0$,

and C' a $(G-1)$ by G matrix, each row of which is a set of coefficients for a between-groups contrast. The rows of C' must be linearly independent, and taken together must imply equality of the mean vectors as in (2.4). Two examples of appropriate C' matrices are

$$(2.5) \quad C'_G = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & -1 \\ 0 & 1 & 0 & \dots & 0 & -1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \end{bmatrix}$$

$$\text{and } C'_G = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \end{bmatrix}$$

There are many other equivalent matrices that could be used. The test of this hypothesis is a MANOVA test on Groups with Measures as the multiple dependent variates. Assumptions are those stated in conjunction with (1.1) but with no restrictions.

A univariate test on Groups can be performed from the multivariate model by using an appropriate A matrix to sum over Measures or to take means over Measures. The C' and Γ matrices are the same as those in the multivariate test. The A matrix for sums is the transpose of

$$(2.6) \quad A_G' = (1 \quad 1 \quad 1 \quad \dots \quad 1)$$

so that

$$(2.7) \quad UA_G = \begin{bmatrix} \mu_1(1) + \mu_2(1) + \dots + \mu_M(1) \\ \mu_1(2) + \mu_2(2) + \dots + \mu_M(2) \\ \vdots \\ \mu_1(G) + \mu_2(G) + \dots + \mu_M(G) \end{bmatrix} = \begin{bmatrix} \mu_{+(1)} \\ \mu_{+(2)} \\ \vdots \\ \mu_{+(G)} \end{bmatrix}$$

so that $C_G' UA_G = \Gamma$, using any appropriate C' matrix is easily shown to be equivalent to

$$(2.8) \quad H_G: \mu_{+(1)} = \mu_{+(2)} = \dots = \mu_{+(G)}$$

which is precisely the hypothesis discussed in Zalar's univariate paper, and the test is also equivalent. If we use means over measures we select the transpose of

$$(2.9) \quad A_G' = (1/M \quad 1/M \quad 1/M \quad \dots \quad 1/M)$$

which yields

$$(2.10) \quad H_G: \mu_{.(1)} = \mu_{.(2)} = \dots = \mu_{.(G)}$$

which is again the hypothesis discussed by Zalar in the univariate paper. Note that the A matrices in (2.6) and (2.9) can be thought of

as reducing the parameter matrix to a column vector which reduces the problem to a univariate one. In this case S_h and S_e are the scalar sums of squares and the univariate F-ratio is formed from them. The assumptions underlying the hypothesis test are those discussed in the univariate paper. If A_G is normalized the test statistics remain invariant but the sums of squares and mean squares reflect the normalization. This is the approach used by BMDP2V.

2.2 Tests on Measures and Measures by Groups Interaction

The multivariate test on Measures is a test of

$$(2.11) \quad H_{M(\text{mult.})}: \begin{bmatrix} \mu_1(1) \\ \mu_1(2) \\ \vdots \\ \mu_1(G) \end{bmatrix} = \begin{bmatrix} \mu_2(1) \\ \mu_2(2) \\ \vdots \\ \mu_2(G) \end{bmatrix} = \dots = \begin{bmatrix} \mu_M(1) \\ \mu_M(2) \\ \vdots \\ \mu_M(G) \end{bmatrix}$$

To express this hypothesis in the form (1.3) the roles of the C' and A matrices are the reverse of those for the test on Groups. Thus we choose $C' = I_G$, $\Gamma = 0$, and A such that the columns express contrasts between the columns of U . Two examples of appropriate A matrices are

$$(2.12) \quad A_M = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \\ -1 & -1 & \dots & -1 \end{bmatrix} \quad \text{and} \quad A_M = \begin{bmatrix} 1 & 0 & \dots & 0 \\ -1 & 1 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & -1 \end{bmatrix}$$

As described in the univariate paper there are two univariate

tests on measures (unless the group sizes are all identical, in which case the two are identical). The "unweighted" test on Measures uses a C' matrix to take an unweighted average over groups,

$$(2.13) \quad C_M' = \begin{pmatrix} 1/G & 1/G & \dots & 1/G \end{pmatrix}$$

and any appropriate A matrix. Using (2.13) yields

$$(2.14) \quad C_M' U = \begin{bmatrix} \frac{1}{G} \sum \mu_1(g) & \frac{1}{G} \sum \mu_2(g) & \dots & \frac{1}{G} \sum \mu_M(g) \\ \mu_1(.) & \mu_2(.) & \dots & \mu_M(.) \end{bmatrix}$$

Postmultiplication by an appropriate A matrix (eg., either matrix in (2.12)) and using $\Gamma = 0$ results in the hypothesis

$$(2.15) \quad H_{M(\text{univ})}: \mu_1(.) = \mu_2(.) = \dots = \mu_M(.)$$

Alternatively we can use

$$(2.16) \quad C_{M*}' = \begin{bmatrix} \frac{n_1}{\sum n_g} & \frac{n_2}{\sum n_g} & \dots & \frac{n_G}{\sum n_g} \end{bmatrix}$$

with an appropriate A matrix and we have a "weighted" hypothesis similar to that discussed in the univariate paper. Then

$$(2.17) \quad C_{M*}' U = \begin{bmatrix} \frac{1}{\sum n_g} (\sum n_g \mu_1(g)) & \frac{1}{\sum n_g} (\sum n_g \mu_2(g)) & \dots & \frac{1}{\sum n_g} (\sum n_g \mu_M(g)) \\ \bar{\mu}_1(.) & \bar{\mu}_2(.) & \dots & \bar{\mu}_M(.) \end{bmatrix}$$

Comparison of (2.16) with (2.13) shows that the "unweighted" and "weighted" hypotheses become identical when all the n_g are equal.

The univariate tests on Measures described above are not equivalent to the tests described in the univariate paper because the orthonormal transformation has not been incorporated. As was mentioned by Zalar one must use orthonormal contrasts as a basis for transformations in order that the data meet the univariate assumptions. In the

multivariate model the A matrix in the hypothesis is essentially a transformation matrix. If we let A^* be the A matrix after orthonormalization, we can think of this matrix transforming the model from that given in (1.1) to

$$(2.18) \quad YA^* = WUA^* + EA^*$$

which we may write as

$$(2.19) \quad Y^* = WU^* + E^*$$

and from (2.19) we can test hypotheses on the parameters of these transformed variates. If we choose the columns of A to be orthogonal, as discussed in the univariate paper, then A^* is simply a normalization of A. If we choose as an A matrix one of those in (2.12), which do not have orthogonal columns, then an orthonormalization must be performed (see, for example, Timm, 1975, pp. 12-15) on A in order to recover the univariate hypothesis test statistic from the multivariate analysis. As was mentioned by Zalar in the univariate paper, if the assumption on E (stated below (1.1)) is met, the orthonormalization of A insures that the assumption is satisfied for E^* in (2.19).

In order to recover the univariate statistics one computes the univariate sums of squares from S_e^* and S_h^* , the error and hypothesis sums of squares and products matrices when A^* is used in the hypothesis. The univariate sums of squares are

$$(2.20) \quad SS_h = \text{Tr}(S_h^*) \quad \text{and} \quad SS_e = \text{Tr}(S_e^*)$$

where Tr represents the trace, or sum of the diagonal elements of a matrix. This procedure is outlined by Bock (1975, pp. 463, 479) ^{and Timm and Carlson (1973)}. The mean squares and F-ratios are computed as usual from the sums of squares in (2.20).

In order to test a Groups by Measures interaction one simply combines a C' matrix for group differences with an A matrix for a test of

Measures. Thus a multivariate test of the interaction is performed by choosing C_G' in (2.5) and A_M in (2.12). In order to compute the univariate test statistic one combines C_G' with the orthonormalized A_M^* , and computes the sums of squares from the traces of the sums of squares and products matrices as in the tests on Measures. The FRMLM and MULTIVARIANCE (Finn, 1976) computer programs will perform the orthonormalizations and compute the univariate statistics for the user.

3.0 The Group by Measures Design with One Covariate Measure

The basic model may be written as (1.1) with Y, as in Section 2, containing the M measures observed on N persons. The Design matrix is

$$(3.1) \quad W = \begin{bmatrix} 1_{n_1} & 0_{n_1} & \dots & 0_{n_1} & x_{n_1} \\ 0_{n_2} & 1_{n_2} & \dots & 0_{n_2} & x_{n_2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0_{n_G} & 0_{n_G} & \dots & 1_{n_G} & x_{n_G} \end{bmatrix}$$

where x_{n_g} is the n_g -element vector of values on the covariate in the g^{th} group. The parameter matrix is

$$(3.2) \quad U = \begin{bmatrix} a_1(1) & a_2(1) & \dots & a_M(1) \\ a_1(2) & a_2(2) & \dots & a_M(2) \\ \vdots & \vdots & & \vdots \\ a_1(G) & a_2(G) & \dots & a_M(G) \\ \beta_1(.) & \beta_2(.) & \dots & \beta_M(.) \end{bmatrix}$$

where $\alpha_m(g)$ is the intercept for the regression line for the m^{th} Measure regressed on X in the g^{th} Group, and $\beta_m(.)$ is the slope for the m^{th} Measure regressed on X . The dot subscript indicates that the slope parameters are assumed homogeneous across groups. If the covariate values are entered as deviations from the grand mean the intercept parameters become the parameters estimated by the "adjusted means" in the usual ANOVA terminology. We prefer to refer to them as intercepts since they really are regression intercept parameters and the covariate values need not be centered in order to perform the analysis.

3.1 The Test on Groups

A between-Groups test can be performed using a C' matrix as in (2.5) but with a column of zeros appended in order to keep the β 's out of the hypothesis. We may use, for example,

$$(3.3) \quad C'_G = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 & 0 \end{bmatrix}$$

If the A matrix for this hypothesis is I_M the test is multivariate and tests a hypothesis similar to (2.4) in the no-covariate design. This test is the one that is performed by commonly-used computer programs such as MULTIVARIANCE (Finn, 1976). It should be used very carefully since it involves fitting a model with a different estimate of the slope parameter for each measure. The basic model does not implicitly incorporate an assumption of equal slopes over measures, but the procedure does involve summing or averaging over measures. In view of this fact it seems obvious that we must assume equality of

these slope parameters whenever we use the technique.

In order to recover the usual univariate ANCOVA test on Groups we must, as in the case of no covariates, use an A matrix that sums or averages over Groups. Hence if we choose (2.6) or (2.9) as an A matrix we can perform the univariate test on Groups.

A model that would seem to be more appropriate is one that explicitly incorporates the assumption of homogeneity of slopes over Measures. In order to specify such a model we need to specify a restriction of a different form,

$$(3.4) \quad R'UT = \textcircled{H}$$

In Section 2 a restricted model was discussed with $T = A$, where A is the post matrix in the hypothesis. In the model under consideration here the A matrix used to test the hypothesis of no Group differences would be an identity matrix in the multivariate case. The matrices in the restriction would be

$$(3.5) \quad \begin{aligned} R' &= (0 \quad 0 \quad . \quad . \quad . \quad 1) \\ T &= \begin{bmatrix} 1 & 0 & . & . & . & 0 \\ -1 & 1 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 1 \\ 0 & 0 & . & . & . & -1 \end{bmatrix} \end{aligned} \quad \textcircled{H} = 0$$

so that $R'UT = \textcircled{H}$ is equivalent to $\beta_1(.) = \beta_2(.) = \dots = \beta_M(.)$.

Then testing the hypothesis with C_G' in (3.3) and $A = I_M$ would be a multivariate test assuming equality of slopes over measures. Such a restricted model is discussed by Carlson and Timm (1980) but has not yet been fully developed. It is related to that of Kabe (1975).

A univariate test could conceivably be performed with A as in (2.6) or (2.9). Such a test would not be equivalent to the usual univariate tests.

3.2 Tests on Measures and Groups by Measures

The tests of these hypotheses are analogous to those in Section 2.2, incorporating the covariate as in the last section. If we use an unrestricted model, as is common practice, the basic model we are fitting yields different slope estimates for each measure. This may be seen by examining the parameter matrix, U, in (3.2). In the multivariate test of Measures we may, for example, use

$$(3.6) \quad C_M' = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix} \quad A_M = \begin{bmatrix} 1 & 0 & \dots & 0 \\ -1 & 1 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & -1 \end{bmatrix}$$

$$\Gamma = 0$$

As in the case of no covariates the univariate statistics can be computed by orthonormalizing the A matrix and computing the sums of squares as traces of the matrices of sums of squares and products. The multivariate test of no Groups by Measures interaction could use C_G' in (3.3) with A_M in (3.6). Again the univariate test is recovered by orthonormalization.

One consequence of using this unrestricted model is the fact that the multivariate analysis is not consistent with the usual univariate analysis. Ceurvorst and Stock (1978) point out that in the multivariate analysis there is an adjustment for the covariate in the Measures and Groups by Measures tests. However, as is pointed

out by Winer (1971) and Zalar in the univariate paper, in the usual univariate analysis there is no adjustment on these effects. The reason for this inconsistency, as mentioned above, is that the multivariate model as usually formulated (and as performed by the unrestricted FRMLM, and by MULTIVARIANCE, which Ceurvorst and Stock used) involves testing for no differences in intercepts for the different measures without explicitly incorporating an assumption of equal slopes.

The hypothesis of no Measures effect is

$$(3.7) \quad H: \begin{bmatrix} \alpha_1(1) \\ \alpha_1(2) \\ \vdots \\ \alpha_1(G) \end{bmatrix} = \begin{bmatrix} \alpha_2(1) \\ \alpha_2(2) \\ \vdots \\ \alpha_2(G) \end{bmatrix} = \dots = \begin{bmatrix} \alpha_M(1) \\ \alpha_M(2) \\ \vdots \\ \alpha_M(G) \end{bmatrix}$$

in the presence of the slope parameters in the last row of (3.2), which are not specified to be homogeneous. Note that in forming the C' matrix for this hypothesis we use an identity matrix to which a column of zeros has been appended in order to keep the β 's out of the hypothesis, as in (3.6).

The hypothesis of no Groups by Measures interaction, similarly,

is

$$(3.8) \quad H: \begin{bmatrix} \alpha_1(1) - \alpha_2(1) \\ \alpha_1(2) - \alpha_2(2) \\ \vdots \\ \alpha_1(G) - \alpha_2(G) \end{bmatrix} = \begin{bmatrix} \alpha_2(1) - \alpha_3(1) \\ \alpha_2(2) - \alpha_3(2) \\ \vdots \\ \alpha_2(G) - \alpha_3(G) \end{bmatrix} = \dots = \begin{bmatrix} \alpha_K(1) - \alpha_M(1) \\ \alpha_K(2) - \alpha_M(2) \\ \vdots \\ \alpha_K(G) - \alpha_M(G) \end{bmatrix}$$

$$K = M - 1$$

which is a hypothesis of equal distances between the regression lines assuming equal slopes (over groups but not over measures).

When the univariate analysis on Measures is performed, using a C' matrix to sum or average over Groups, the test implicitly assumes no Measures by Groups Interaction. If there truly is an interaction one is averaging over parameters that are not equal. Thus many writers suggest not testing the univariate Measures effect if the interaction hypothesis is rejected. It is always possible (but not known) of course, that a type one error is made on the interaction test and the test on Measures is valid. The ideal situation would be to know a priori that the interaction does not exist.

An alternative analysis is to use the restricted model developed by Carlson and Timm (1980). Using the restricted FRMLM, incorporating (1.2) with, for example:

$$(3.9) \quad R' = (0 \ 0 \ \dots \ 1) \quad A = \begin{bmatrix} 1 & 0 & \dots & 0 \\ -1 & 1 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & 1 \\ 0 & 0 & & -1 \end{bmatrix}$$

$$\textcircled{H} = 0$$

we estimate

$$(3.10) \quad U_r = \begin{bmatrix} \alpha_1^r(1) & \alpha_2^r(1) & \dots & \alpha_M^r(1) \\ \alpha_1^r(2) & \alpha_2^r(2) & \dots & \alpha_M^r(2) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^r(G) & \alpha_2^r(G) & \dots & \alpha_M^r(G) \\ \beta_r & \beta_r & \dots & \beta_r \end{bmatrix}$$

where β_r is a slope parameter restricted to equality over measures and $\alpha_{m(g)}^r$ is a restricted intercept parameter. Although β_r appears M times there is really only one slope parameter. The estimate of U_r is that shown in (1.6). The restricted sums of squares and products matrices for each hypothesis are computed using (1.3), (1.8) and (1.9). The C' and A matrices for the hypotheses are the same as those shown above for the unrestricted model. As usual the univariate statistics may be recovered by using orthonormalized A matrices. When this restricted model is employed there is no adjustment for the covariate and the results are consistent with the usual univariate analysis. The writer feels that this analysis is more appropriate than that performed using the unrestricted model with the inherent inconsistency with the usual univariate analysis.

3.3 Example and Discussion

As an example the artificial data analyzed in the univariate paper are used. In this dataset $G = 3$, $M = 4$, and the sample groups are of unequal sizes: $n_1 = 5$, $n_2 = 7$, $n_3 = 6$.

Using the unrestricted model the parameter estimates are

$$(3.11) \quad \hat{U} = \begin{bmatrix} 1.396 & 5.252 & 3.945 & 4.541 \\ 1.907 & 6.800 & 5.879 & 5.083 \\ 3.765 & 6.698 & 5.838 & 6.423 \\ .848 & -.482 & .175 & -.208 \end{bmatrix}$$

Using this unrestricted model we can test for equality of the slopes across measures by using R' and A in (3.9) as the C' and A matrices. For this test the sums of squares and products matrices are

$$S_h = \begin{bmatrix} 47.522 & & (\text{symm.}) \\ -23.473 & 11.594 & \\ 13.693 & -6.763 & 3.945 \end{bmatrix}$$

$$(3.12) \quad S_e = \begin{bmatrix} 79.940 & & (\text{symm.}) \\ -54.041 & 88.463 & \\ 13.607 & -36.437 & 56.755 \end{bmatrix}$$

and, using Wilks' lambda criterion,

$$(3.13) \quad \Lambda = .596 > U_{(3,1,14)}^{.05} = .534$$

so the hypothesis of homogeneity is not rejected at the .05 level. Hence we conclude that the variation in slopes across the 4 measures (the estimates are the last row of U in (3.11)) is due to sampling fluctuation.

Although, as was stated above, the writer does not believe it meaningful to test for equality of intercepts in the unrestricted model the univariate tests are reported here for purposes of discussion. For each test reported below the multivariate model was used with an orthonormal A matrix and the univariate sums of squares were computed as the traces of the resulting matrices of sums of squares and products (the FRMLM computer program performs the orthonormalization upon request by the user and computes the univariate statistics whenever the orthonormalization is specified). Comparison with the analysis in Zalar's univariate paper will show that the test on Groups is identical to that given by univariate programs (BMDP2V, FRULM). The Measures and Groups by Measures interaction tests are, however, different. This difference is due to the failure to explicitly incorporate the homogeneity of slopes assumption.

	Hypothesis	Sum Of Squares	df	MS	F	p-level
a)	<u>Groups</u>					
	Between	39.159	2	19.579	12.212	.0009
	Error	22.445	14	1.603		
b)	<u>Measures (unweighted)</u>					
	Measures	23.725	3	7.908	3.095	.0370
	Error	107.310	42	2.555		
(3.14)	c)	<u>Measures (weighted)</u>				
	Measures *	24.036	3	8.012	3.136	.0353
	Error	107.310	42	2.555		
	d)	<u>Groups by Measures</u>				
	G by M	10.351	6	1.725	0.675	.6703
	Error	107.310	42	2.555		

In order to incorporate an assumption of equal slopes over measures the data were analyzed using a restricted model. The R' and A matrices in (3.9) were used. The restricted estimate of the parameter matrix was

$$(3.15) \quad \hat{U}_r = \begin{bmatrix} 3.384 & 3.784 & 4.184 & 3.784 \\ 4.310 & 5.024 & 6.167 & 4.167 \\ 5.931 & 5.098 & 6.098 & 5.598 \\ .083 & .083 & .083 & .083 \end{bmatrix}$$

Note that the last row on this matrix contains only one unique slope estimate since we are specifying equal slopes. The value, .083, is exactly the same as that computed by the univariate computer programs (FRULM, BMDP2V) as reported in the univariate paper. The results of the univariate tests on this model, based on traces of the sums of squares and products matrices with orthonormal A matrices are shown below.

	Hypothesis	Sum Of Squares	df	MS	F	p-level
a)	<u>Measures (unweighted)</u>					
	Measures	11.316	3	3.772	1.266	.2976
	Error	134.114	45	2.980		
b)	<u>Measures (weighted)</u>					
(3.16)	Measures *	13.264	3	4.421	1.484	.2318
	Error	134.114	45	2.980		
c)	<u>Groups by Measures</u>					
	G by M	9.372	6	1.562	0.524	.7870
	Error	134.114	45	2.980		

Comparison with the analyses reported in Zalar's univariate paper shows that the tests are identical to those given by BMDP2V and FRULM. As currently implemented the multivariate program, FRMLM, cannot compute a valid test on Groups which incorporates the equal-slopes assumption since it requires that the A matrix in the restriction be identical to the A matrix in the hypothesis. As previously mentioned we are currently attempting to develop a model which will allow such an analysis.

The multivariate test on Groups, using the unrestricted model, yielded the following results on Wilks' lambda criterion:

$$(3.17) \quad \Lambda = .255 < U_{(4,2,14)}^{.05} = .314$$

so the hypothesis of no Group differences is rejected at the .05 level.

The multivariate tests on Measures and Groups by Measures interaction yielded the following results on Wilks' lambda criterion:

$$(3.18) \quad \Lambda = .548 > U_{(3,3,15)}^{.05} = .309 \quad (\text{Measures})$$

$$\Lambda = .759 > U_{(3,2,15)}^{.05} = .405 \quad (\text{Measures by Groups})$$

Thus these two hypotheses cannot be rejected at the .05 level.

In comparing the various results reported for these data it should be noted that choice of method of analysis determines whether some of the hypotheses are, or are not, rejected. The univariate measures hypothesis is rejected at the .05 level if the unrestricted model is used but not if the restricted model is selected. The restricted univariate results are, however, in accordance with the restricted multivariate results. Although this accordance is nice it should be remembered that it does not always occur. At any rate, as was mentioned above it is the writer's belief that the restricted model should be used whether univariate or multivariate procedures are used. Unfortunately this is not currently possible for the test on Groups.

4.0 The Groups by Measures Design with M Covariate Measures

When the repeated measures factor in a design involves experimental treatments the researcher may choose to take measurements on the covariate prior to the administration of each treatment. In this case there are M covariate measures, one associated with each of the M measures. Although the analysis of such a design is fairly easy when using a univariate analysis (as discussed in Zalar's univariate paper), problems are encountered in formulating an appropriate multivariate linear model approach.

We may formulate the model as in (1.1) with Y, as in Section 2 containing the M measures on N persons. The design and parameter matrices are similar to those in (3.1) and (3.2) except that there are M columns of covariates in W and M rows of slope parameters in U, as shown below.

$$(4.1) \quad W = \begin{bmatrix} 1_{n_1} & 0_{n_1} & \dots & 0_{n_1} & x_{1n_1} & x_{2n_1} & \dots & x_{Mn_1} \\ 0_{n_2} & 1_{n_2} & \dots & 0_{n_2} & x_{1n_2} & x_{2n_2} & \dots & x_{Mn_2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0_{n_G} & 0_{n_G} & \dots & 1_{n_G} & x_{1n_G} & x_{2n_G} & \dots & x_{Mn_G} \end{bmatrix}$$

$$(4.2) \quad U = \begin{bmatrix} a_1(1) & a_2(1) & \dots & a_M(1) \\ a_1(2) & a_2(2) & \dots & a_M(2) \\ \vdots & \vdots & & \vdots \\ a_1(G) & a_2(G) & \dots & a_M(G) \\ \beta_{11} & \beta_{21} & \dots & \beta_{M1} \\ \beta_{12} & \beta_{22} & \dots & \beta_{M2} \\ \vdots & \vdots & & \vdots \\ \beta_{1M} & \beta_{2M} & \dots & \beta_{MM} \end{bmatrix}$$

where β_{km} is the slope parameter for the regression of the k^{th} dependent variate onto the m^{th} covariate, assumed constant across groups. The problem with this formulation of the model is that we really wish to use the m^{th} covariate only with the m^{th} dependent variate. Hence we need to incorporate the assumption

$$(4.3) \quad \beta_{km} = 0, \quad k \neq m$$

into the model. That is, we require a way of specifying restrictions such that the restricted parameter matrix has the form

$$(4.4) \quad U_r = \begin{bmatrix} a_1(1) & a_2(1) & \cdot & \cdot & \cdot & a_M(1) \\ a_1(2) & a_2(2) & \cdot & \cdot & \cdot & a_M(2) \\ \vdots & \vdots & & & & \vdots \\ \vdots & \vdots & & & & \vdots \\ \vdots & \vdots & & & & \vdots \\ \beta_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \beta_2 & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & & & & \vdots \\ \vdots & \vdots & & & & \vdots \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & & & & \beta_M \end{bmatrix}$$

Unfortunately there is no way of specifying such a restriction in the form $R'UT = \textcircled{H}$. Hence the multivariate linear model considered in this presentation may not be used to analyze data in the case of M covariate measures with M repeated measures unless the researcher wishes to use the unrestricted model. In using that model all M covariates are used with each dependent variate. An analysis using such a model is not likely to be appropriate in most cases and is not equivalent to the usual univariate analysis. Thus the writer would recommend that the univariate model be used to analyze data for this type of design.

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